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# A Reciprocity Theorem for the Interaction of Electromagnetic Plane Waves with a One-Dimensional Inhomogeneous Slab*

The geometry of the problem is shown in Fig. 1. A plane wave, varying in time as $e^{-j\omega t}$, with components $E_{i\|}$ and $E_{iy}=E_{i\perp}$ is incident at an angle $\theta$ upon an inhomogeneous slab at the boundary $z=0$. The complex index of refraction of the medium, $n(z)$ and its derivative varies continuously in a direction normal to the boundaries. $E_{r\|}$ and $E_{ry}$ are reflected components, and $E_{t\|}$ and $E_{ty}$ are transmitted components of the electric vector. Comparison of these solutions with those generated by inverting the slab will establish a reciprocity criteria. This inversion procedure is equivalent to analyzing two-way transmission through a given inhomogeneous slab.

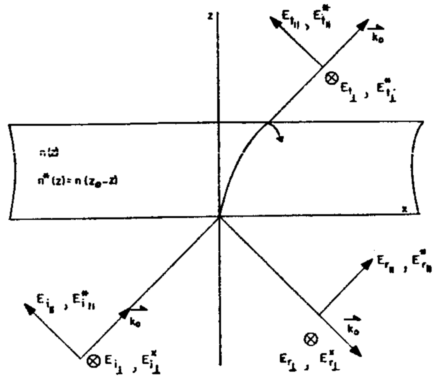


Fig. 1.

Stratton[1] has shown that the solutions to the wave equation are uniquely described by those components of the electric vector which are perpendicular to and parallel to the plane of incidence, which is that plane containing the propagation vector, $\boldsymbol{k}_0$ and the vector normal to the boundaries of the slab. Since the parallel component of the electric vector is related to the $y$ component of the magnetic vector, $H_y$, the properties are completely defined by the wave equations for $E_y$ and $H_y$. These equations for a medium which is inhomogeneous along $z$ are

$$\frac{d^2E_y(z)}{dz^2}+k_0^2[n^2(z)-\sin^2\theta]E_y(z)=0 \qquad (1)$$

$$\frac{d^2H_y(z)}{dz^2}-\frac{1}{n^2(z)}\frac{dn^2(z)}{dz}\frac{dH_y(z)}{dz}+k_0^2[n^2(z)-\sin^2\theta]H_y(z)=0. \qquad (2)$$

The solutions to these equations are of the form

$$\frac{E_y}{E_{0y}}=C_1u_1(z)+C_2u_2(z);$$

$$\frac{H_y}{H_{0y}}=C_3v_1(z)+C_4v_2(z), \qquad (3)$$

where $E_{0y}$ and $H_{0y}$ are the amplitudes of the incident $E$ and $H$ vectors and the $C$'s are constants. Continuity of the tangential components of $E$ and $H$ at $z=0$ and $z=z_0$, lead to the following boundary conditions for (1) and (2):

$$\begin{bmatrix} u_1(0) & u_2(0) & -1 & 0 \\ u_1'(0) & u_2'(0) & jk_0\cos\theta & 0 \\ u_1(z_0) & u_2(z_0) & 0 & -e^{jk_0z_0\cos\theta} \\ u_1'(z_0) & u_2'(z_0) & 0 & -jk_0\cos\theta e^{jk_0z_0\cos\theta} \end{bmatrix}\begin{Bmatrix} C_1 \\ C_2 \\ R_\perp \\ T_\perp \end{Bmatrix}=\begin{Bmatrix} 1 \\ jk_0\cos\theta \\ 0 \\ 0 \end{Bmatrix} \qquad (4)$$

$$\begin{bmatrix} v_1(0) & v_2(0) & -1 & 0 \\ v_1'(0) & v_2'(0) & jk_0n^2(0)\cos\theta & 0 \\ v_1(z_0) & v_2(z_0) & 0 & -e^{jk_0z_0\cos\theta} \\ v_1'(z_0) & v_2'(z_0) & 0 & -jk_0n^2(z_0)\cos\theta e^{jk_0z_0\cos\theta} \end{bmatrix}\begin{Bmatrix} C_3 \\ C_4 \\ R_\| \\ T_\| \end{Bmatrix}=\begin{Bmatrix} 1 \\ jk_0n^2(0)\cos\theta \\ 0 \\ 0 \end{Bmatrix}. \qquad (5)$$

The $R$'s and $T$'s are the complex reflection and transmission coefficients, respectively; and $k_0$ is the wave number of the incident wave. A medium whose index of refraction is $n^*(z)$ results in identical boundary conditions except the $R$'s, $T$'s, $C$'s, $u$'s, $v$'s, and $n$'s are superscripted by a star (*). The index of refraction, $n^*(z)$ of a slab which is inverted may be defined in terms of the given value of $n(z)$ such that $n^*(z)=n(z_0-z)$. Consequently, the solutions of the inverted slab become

$$u_1^*(z)=u_1(z_0-z);\quad u_2^*(z)=u_2(z_0-z)$$

$$v_1^*(z)=v_1(z_0-z);\quad v_2^*(z)=v_2(z_0-z), \qquad (6)$$

and the boundary conditions for the starred solutions become

$$\begin{bmatrix} u_1(0) & u_2(0) & 0 & -e^{jk_0z_0\cos\theta} \\ u_1'(0) & u_2'(0) & 0 & jk_0\cos\theta e^{jk_0z_0\cos\theta} \\ u_1(z_0) & u_2(z_0) & -1 & 0 \\ u_1'(z_0) & u_2'(z_0) & -jk_0\cos\theta & 0 \end{bmatrix}\begin{Bmatrix} C_1^* \\ C_2^* \\ R_\perp^* \\ T_\perp^* \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 1 \\ -jk_0\cos\theta \end{Bmatrix} \qquad (7)$$

$$\begin{bmatrix} v_1(0) & v_2(0) & 0 & -e^{jk_0z_0\cos\theta} \\ v_1'(0) & v_2'(0) & 0 & jk_0n^2(0)\cos\theta e^{jk_0z_0\cos\theta} \\ v_1(z_0) & v_2(z_0) & -1 & 0 \\ v_1'(z_0) & v_2'(z_0) & -jk_0n^2(z_0)\cos\theta & 0 \end{bmatrix}\begin{Bmatrix} C_3^* \\ C_4^* \\ R_\|^* \\ T_\|^* \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 1 \\ -jk_0n^2(z_0)\cos\theta \end{Bmatrix}. \qquad (8)$$

Let the determinants of the matrix equations (4), (5), (7), and (8) be $\Delta_1$, $\Delta_2$, $\Delta_1^*$, and $\Delta_2^*$, respectively. Expansion of the determinants shows that $\Delta_1=-\Delta_1^*$ and $\Delta_2=-\Delta_2^*$. The reflection and transmission coefficients can be solved in terms of $\Delta_1$ and $\Delta_2$; however, the following difference relationships are the quantities of interest:

$$\Delta_1(T_\perp-T_\perp^*)=-2jk_0\cos\theta\left\{\begin{vmatrix} u_1(z_0) & u_2(z_0) \\ u_1'(z_0) & u_2'(z_0) \end{vmatrix}-\begin{vmatrix} u_1(0) & u_2(0) \\ u_1'(0) & u_2'(0) \end{vmatrix}\right\} \qquad (9)$$

$$\Delta_2(T_\|-T_\|^*)=-2jk_0\cos\theta\left\{n^2(0)\begin{vmatrix} v_1(z_0) & v_2(z_0) \\ v_1'(z_0) & v_2'(z_0) \end{vmatrix}-n^2(z_0)\begin{vmatrix} v_1(0) & v_2(0) \\ v_1'(0) & v_2'(0) \end{vmatrix}\right\} \qquad (10)$$

$$\Delta_1(R_\perp-R_\perp^*)=-2jk_0\cos\theta e^{jk_0z_0\cos\theta}\left\{\begin{vmatrix} u_1(0) & u_2(0) \\ u_1'(z_0) & u_2'(z_0) \end{vmatrix}-\begin{vmatrix} u_1(z_0) & u_2(z_0) \\ u_1'(0) & u_2'(0) \end{vmatrix}\right\} \qquad (11)$$

$$\Delta_2(R_\|-R_\|^*)=-2jk_0\cos\theta e^{jk_0z_0\cos\theta}\left\{n^2(0)\begin{vmatrix} v_1(0) & v_2(0) \\ v_1'(z_0) & v_2'(z_0) \end{vmatrix}-n^2(z_0)\begin{vmatrix} v_1(z_0) & v_2(z_0) \\ v_1'(0) & v_2'(0) \end{vmatrix}\right\}. \qquad (12)$$

* Received April 19, 1963. After this derivation was completed, it was discovered that a reciprocity relationship was stated for a plane wave normally incident upon an $N$-layer dielectric (R. Hollis, "Reciprocal relationships in an $n$-slab dielectric," PROC. IRE (*Correspondence*), vol. 49, p. 1579; October, 1961). This paper presents a more general picture by considering continuous variation of the material properties and angle of incidence.

[1] J. A. Stratton, "Electromagnetic Theory," McGraw-Hill Book Company, Inc., New York, N. Y. p. 492; 1941.

From Abel's formula,[2] the Wronskians of (1) and (2) are, respectively

$$\begin{vmatrix} u_1(z) & u_2(z) \\ u_1'(z) & u_2'(z) \end{vmatrix} = \text{const.};$$

$$\begin{vmatrix} v_1(z) & v_2(z) \\ v_1'(z) & v_2'(z) \end{vmatrix} = \text{const.}$$

$$\times \exp\left(\int_0^z d[\ln n^2(z)]\right)$$

$$= \text{const.} \times \frac{n^2(z)}{n^2(0)}. \qquad (13)$$

Therefore,

$$T_\perp = T_\perp^* \qquad T_\| = T_\|^* \qquad (14)$$

$$R_\perp = R_\perp^* \quad \text{if} \begin{vmatrix} u_1(0) & u_2(0) \\ u_1'(z_0) & u_2'(z_0) \end{vmatrix} = \begin{vmatrix} u_1(z_0) & u_2(z_0) \\ u_1'(0) & u_2'(0) \end{vmatrix}$$

$$R_\| = R_\|^* \quad \text{if } n^2(0) \begin{vmatrix} v_1(0) & v_2(0) \\ v_1'(z_0) & v_2'(z_0) \end{vmatrix} = n^2(z_0) \begin{vmatrix} v_1(z_0) & v_2(z_0) \\ v_1'(0) & v_2'(0) \end{vmatrix}. \qquad (15)$$

Thus, at a given frequency, angle of incidence, and polarization, the transmission of electromagnetic waves through a one-dimensional inhomogeneous plasma is independent of the direction of transmission. On the other hand, the reflection coefficients, in general, do not necessarily conform to a reciprocity relationship. These results were checked by running a few cases on a computing machine program developed at Langley.

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[2] F. B. Hildebrand, "Advanced Calculus for Engineers," Prentice-Hall, Inc., Englewood Cliffs, N. J., pp. 31–35; 1957.

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